

Physics of Racing, Part 22:

The Magic Formula: Lateral Version

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In this installment, we review the other side of the magic formula: the one that computes lateral or cornering forces from slip angles (or grip angles). This formula is sufficiently similar to the longitudinal version of Part 21 that we can skip many preliminaries. But it's sufficiently different as to require careful exposition, leading us to define coordinate frames that will serve us throughout the rest of the *Physics of Racing* series. This installment will be one to keep on hand for future reference.

Diving right in, just like its longitudinal sibling, this formula requires some magical constants, fifteen of them this time. Again, from Genta's possible-Ferrari data sheet:

a_0	1.799	dimensionless	a_7	1	dimensionless
a_1	0	1/MN	a_8	0	dimensionless
a_2	1688	1/Kilo	a_9	-6.111/1000	Degree/KN
a_3	4140	N	a_{10}	-3.224/100	Degree
a_4	6.026	KN	$a_{11,1}$	0	1/MN-Degree
a_5	0	1/Degree	$a_{11,2}$	0	1/KiloDegree
a_6	-0.3589	KN	a_{12}	0	1/Kilo
			a_{13}	0	N

where N is Newton, KN is KiloNewton, and MN is MegaNewton. As with the longitudinal magic formula, there are lots of zeros in this particular sample case, but let us not confuse particulars with generalities. The formula can account for much more general cases.

The first helper is the peak, lateral friction coefficient $\mu_{yp} = a_1 F_z + a_2$, measured in inverse Kilos if F_z is in KN. Next is $D = \mu_{yp} F_z$, which is a factor *with the form of* the Newtonian model: normal force times coefficient of friction. In our sample, a_1 is zero, so μ_{yp} acts exactly like a Newtonian friction coefficient. In all cases, we should expect $a_1 F_z$ to be much smaller than a_2 so that it will be, at most, a small correction to the Newtonian behavior.

To get the final force, we correct D with the following empirical factor:

$$\sin(\tau), \tau = a_0 \tan^{-1}(\nu)$$

$$\nu = SB + E \left[\tan^{-1}(SB) - SB \right]$$

This has exactly the same form as the empirical correction factor in the longitudinal version, but the component pieces, S , B , and E are different, here.

$$S = \alpha_{\text{degrees}} + a_8 \gamma_{\text{degrees}} + a_9 F_z + a_{10}$$

where α is the slip angle and γ is the camber angle of the wheel. In practice, we must carefully account for the algebraic signs of the camber angles so that the forces make sense at all four wheels. The usual negative camber, by the 'shop' definition, as measured on the wheel-alignment machine, will generate forces in the positive Y-direction on the right-hand side of the car and in the negative Y-direction on the left-hand side of the car. This comment makes much more sense after we've covered coordinate frames, below.

As before, we get B from a product, albeit one of greatly different form

$$Ba_0 D = a_3 \sin \left[2 \tan^{-1} (F_z / a_4) \right] (1 - a_5 |\gamma|)$$

where $|\gamma|$ is the absolute value of the camber angle, that is, a positive number no matter what the sign of γ . This gives

$$B = \frac{Ba_0 D}{a_0 D} = \frac{a_3 \sin \left[2 \tan^{-1} (F_z / a_4) \right] (1 - a_5 |\gamma|)}{a_0 \mu_{yp} F_z}$$

Almost done; include $E = a_6 F_z + a_7$ and sneak in an additive correction for *ply steer* and *conicity*, which we'll leave undefined in this article:

$$S_v = \left[(a_{11,1} F_z + a_{11,2}) \gamma + a_{12} \right] F_z + a_{13}$$

To arrive at the final formula

$$F_y = D \sin \left(a_0 \tan^{-1} \left\{ SB + E \left[\tan^{-1} (SB) - SB \right] \right\} \right) + S_v$$

This form is almost identical—in form—to the longitudinal version of the magic formula. The individual subcomponents are different in detail, however.

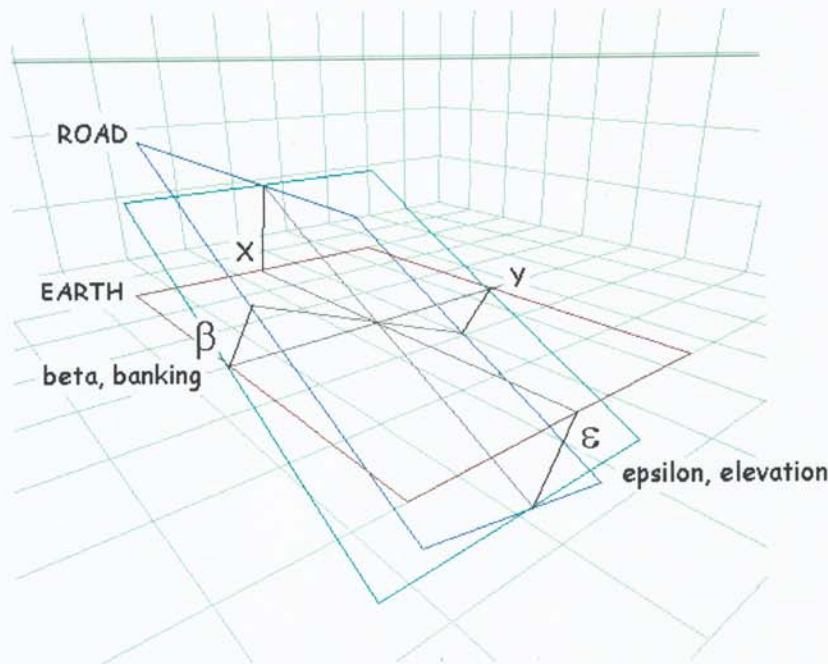
The most important input is the slip angle, α . This is the difference between the actual pathline of the car and the angle of the wheel. To be precise, we must define coordinate systems. We'll stay close to the conventions of the Society of Automotive Engineers (SAE), as published by the Millikens in *Race Car Vehicle Dynamics*. Note that this may differ from some frames we've used in the past, but we're going to stick with this set. There's a lot of intense verbiage in the following, but it's necessary to define precisely what we mean by wheel orientation in all generality. Only then can we measure slip angle as the difference between the path heading of the car and the wheel orientation.

First, is the **EARTH** frame, whose axes we write as $\{X, Y, Z\}$. The Z axis is aligned with Earth's gravitation and points *downward*. The origin of EARTH is fixed w.r.t. the Earth and the X and Y axes point in arbitrary, but fixed, directions. A convenient choice at a typical track might be the center of start/finish with X pointing along the direction of travel of the cars up the main straight. All other coordinate frames ultimately relate back to EARTH, meaning that the location and orientation of every other frame must be given w.r.t. EARTH, directly or indirectly.

The next coordinate frame is **CAR**, whose axes we write as $\{x, y, z\}$. This frame is fixed w.r.t. the sprung mass of the car, that is the body, with x running from tail to nose, y to driver's right, and z downward, roof through seat. Its instantaneous orientation w.r.t. EARTH is the **heading**, ψ . Precisely, consider the line formed by the intersection of EARTH's XY plane with CAR's xz plane. The angle of that line w.r.t. EARTH's X axis is the instantaneous heading of the car. It becomes undefined only when the car points directly up—standing on its tail—or directly down—standing on its nose. To emphasize, **heading is measured in the EARTH frame.**

The next coordinate frame is **PATH**. The velocity vector of the car traces out a curve in 3-dimensional space such that it is tangent to the curve at every instance. The X-direction of PATH points along the velocity vector. The Z-direction of PATH is at right angles to the X direction and in the plane formed by the velocity vector and the Z-direction of EARTH. The Y direction of PATH completes the frame such that XYZ form an orthogonal, right-handed triad. The path of the car lies instantaneously in the XY plane of PATH. PATH ceases to exist when the car stops moving. **Path heading** is the angle of the projection of the velocity vector on EARTH's XY w.r.t. the X-axis of EARTH. Milliken calls this *course angle*, ν (Greek upsilon). Path heading, just like heading, is measured in the EARTH frame. The **sideslip angle of the entire vehicle** is the path heading minus the car heading, $\nu - \psi$. This is positive when the right side of the car slips in the direction of travel.

The next set of coordinate frames is **ROAD_i**, where i varies from 1 to 4; there are four frames representing the road under each wheel, numbered as 1=Left Front, 2=Right Front, 3=Left Rear, 4=Right Rear. Each ROAD_i is located at the force center of its corresponding contact patch at the point $\mathbf{R}_i \equiv (R_i^x, R_i^y, R_i^z)$ w.r.t. EARTH. This point moves with the vehicle, so, more pedantically, the origin of ROAD_i is $\mathbf{R}_i(t)$ written as a function of time. To get the X and Y axes of ROAD_i, we begin with a temporary, flat, coordinate system called **TA_i** aligned with EARTH and centered at \mathbf{R}_i , then elevate by an angle $-90^\circ < \varepsilon < 90^\circ$, to get temporary frame **TB_i**, and bank by an angle $-90^\circ < \beta < 90^\circ$, in that order, as illustrated below:



Consider any point P in space with coordinates $\mathbf{P} \equiv (P^X, P^Y, P^Z)$ w.r.t. EARTH. A little reflection reveals that its location w.r.t. TA_i is $\mathbf{P}_{Ai} \equiv \mathbf{P} - \mathbf{R}_i$, just subtracting coordinates component-by-component. To get coordinates in TB_i , we multiply by the *orthogonal matrix* (once again, see www.britannica.com for brush-up) that does not change the Y components, but increases the Z and decreases the X components of points in the first quadrant for small, positive angles, namely:

$$\begin{pmatrix} \cos \epsilon & 0 & -\sin \epsilon \\ 0 & 1 & 0 \\ \sin \epsilon & 0 & \cos \epsilon \end{pmatrix}$$

We pick this matrix by inspection of the figure above or by application of the right-hand-rule (yup, see britannica) Finally, to bank the system, we need the orthogonal matrix that does not change the X components, but increases the Y and decreases the Z components of first-quadrant points for small, positive angles, namely:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{pmatrix}$$

In case you missed it, we snuck in a reliable, seat-of-the-pants method for getting the signs of orthogonal matrices right. In any event, given \mathbf{P} and \mathbf{R}_i , we compute the coordinates, \mathbf{P}_{Ri} , of the point P in $ROAD_i$ as follows:

$$\begin{pmatrix} \mathbf{P}_{Ri}^X \\ \mathbf{P}_{Ri}^Y \\ \mathbf{P}_{Ri}^Z \end{pmatrix} = \begin{pmatrix} \cos \epsilon & 0 & -\sin \epsilon \\ \sin \beta \sin \epsilon & \cos \beta & \sin \beta \cos \epsilon \\ \cos \beta \sin \epsilon & -\sin \beta & \cos \beta \cos \epsilon \end{pmatrix} \begin{pmatrix} \mathbf{P}^X - \mathbf{R}_i^X \\ \mathbf{P}^Y - \mathbf{R}_i^Y \\ \mathbf{P}^Z - \mathbf{R}_i^Z \end{pmatrix}$$

If the angles are small, $\cos \xi \approx 1$, $\sin \xi \approx \xi$, and the matrix can be simplified to

$$\begin{pmatrix} \mathbf{P}_{Ri}^X \\ \mathbf{P}_{Ri}^Y \\ \mathbf{P}_{Ri}^Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\varepsilon \\ \beta\varepsilon & 1 & \beta \\ \varepsilon & -\beta & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}^X - \mathbf{R}_i^X \\ \mathbf{P}^Y - \mathbf{R}_i^Y \\ \mathbf{P}^Z - \mathbf{R}_i^Z \end{pmatrix}$$

Even at 20 degrees, the errors are only about 6% in the cosine and 2% in the sin, resulting in a maximum error of 12% in the lower right of the matrix. This matrix approximation is suitable for the majority of applications. One feature of orthogonal matrices is that their *inverse* is their *transpose*, that is, the matrix derived by flipping everything about the main diagonal running from upper left to lower right. In the small-angle approximation, we get

$$\begin{pmatrix} 1 & 0 & -\varepsilon \\ \beta\varepsilon & 1 & \beta \\ \varepsilon & -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta\varepsilon & \varepsilon \\ 0 & 1 & -\beta \\ -\varepsilon & \beta & 1 \end{pmatrix} = \begin{pmatrix} 1+\varepsilon^2 & 0 & 0 \\ 0 & 1+\beta^2(1+\varepsilon^2) & \beta\varepsilon^2 \\ 0 & \beta\varepsilon^2 & 1+\beta^2+\varepsilon^2 \end{pmatrix}$$

The right-hand side is very close to the unit matrix because the squares of small angles are smaller, yet. With the inverse matrix we can convert from coordinates in ROAD_{*i*} to coordinates in EARTH:

$$\begin{pmatrix} \mathbf{P}^X \\ \mathbf{P}^Y \\ \mathbf{P}^Z \end{pmatrix} = \begin{pmatrix} 1 & \beta\varepsilon & \varepsilon \\ 0 & 1 & -\beta \\ -\varepsilon & \beta & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}_{Ri}^X \\ \mathbf{P}_{Ri}^Y \\ \mathbf{P}_{Ri}^Z \end{pmatrix} + \begin{pmatrix} \mathbf{R}_i^X \\ \mathbf{R}_i^Y \\ \mathbf{R}_i^Z \end{pmatrix}$$

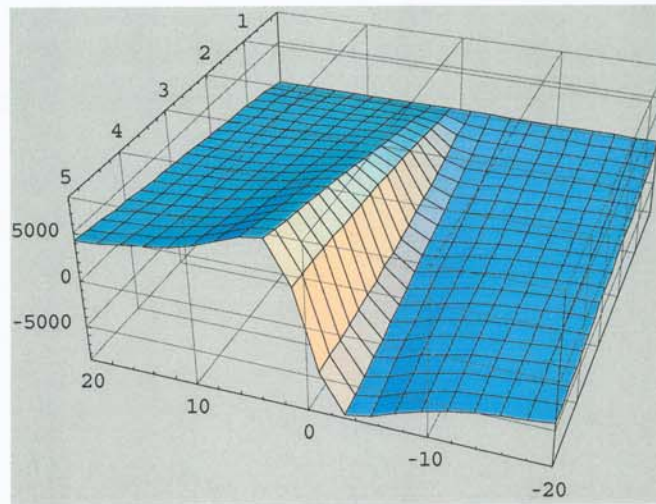
The last set of coordinate frames is **WHEEL_{*i*}**. As with ROAD_{*i*}, there is one instance per wheel. WHEEL_{*i*} is centered at the wheel hub. Under normal rolling, the coordinates of its origin in ROAD_{*i*} are $\mathbf{W}_{Ri} \equiv (0, 0, -R_i)$, where R_i is the loaded radius of the tire-wheel combination. Pedantically, R_i should be corrected for elevation and banking, but such corrections would be small for ordinary angles—on the order of $2 - \cos \beta \cos \varepsilon$ —plus it seems *not* to be standard practice (I can find no reference to it in my sources). More important is the orientation of WHEEL_{*i*}. Consider the plane occupied by the wheel itself. This plane intersects ROAD_{*i*} in a line that defines the X direction of WHEEL_{*i*}, with the positive direction being as close to that of travel as possible. The Y direction points to driver's right. The wheel plane is tilted by a camber angle, γ , about the X-axis of the WHEEL coordinate system. To emphasize: WHEEL_{*i*} **does not include wheel camber**, and it differs from ROAD only by a rotation about ROAD's Z axis that accounts for the pointing direction of the wheel.

At this point, you should create a mental picture of these coordinate frames under typical racing conditions. Picture a CAR frame yawed at some heading w.r.t. EARTH—and perhaps pitched and rolled a bit; a PATH frame aligned at some slightly different path heading; and individual ROAD and WHEEL frames under each tire contact patch, where the ROAD frames are perhaps tilted a bit w.r.t. EARTH and the WHEEL frames are aligned with the wheel planes but coplanar with the ROAD frames. For a car traveling on a flat road at a stable, flat attitude, the XY planes of CAR, PATH, and EARTH would all coincide and

would differ from one another only in the yaw angles ψ and ν . When some tilting is engaged, ψ and ν are still defined by the precise projection mechanisms explained above.

Now, imagine the X-axis of CAR projected on the XY plane of each WHEEL frame and translated—without changing its direction—to the origin of WHEEL. The angle of WHEEL's X axis, which is the same as the plane containing the wheel, w.r.t. the projection of CAR's X axis, defines the steering angle, δ , of that wheel. Finally, imagine PATH's X axis projected onto the XY plane of WHEEL in exactly the same way. Its angle w.r.t. to the X axis of WHEEL, in all generality, defines the slip angle. Since WHEEL is tilted w.r.t. gravitational *down*, the load, F_z , on the contact patch, which we need for the magic formula, must be computed in WHEEL. It will be smaller than the total weight, W_i , by factors of $\cos \beta$ and $\cos \varepsilon$, which are obviously unity under the small-angle approximation.

At last, we can plot the magic formula:



The horizontal axis measures slip angle, in degrees. The vertical axis measures lateral, cornering force, in Newtons. The deep axis measures vertical load on the contact patch, in KiloNewtons. We can see that these tires have a peak at about 4 degrees of slip and that cornering force goes *down* as slip goes up on either side of the peak. On the high side of the peak, we have dynamic understeer, where turning the wheel more makes the situation worse. This is a form of instability in the control system of car and driver.

As a final comment, let me say that I am somewhat dismayed that the magic formula does *not* account for any variation of the lateral force with speed. Intuitively, the forces generated at high speeds must be greater than the forces at low speed with the same slip angles. However, the literature—sometimes explicitly, and sometimes by sin of omission—states that the magic formula doesn't deal with it. One of the reasons is that, experimentally, effects of speed are extremely difficult to separate from effects of temperature. A fast-moving tire becomes a hot tire very quickly on a test rig. Another reason is that theoretical data is usually closely guarded and is not likely to make it into a consensus approximation like the magic formula. This is a fact of life that we hope will not affect our analyses too adversely from this point on.