The Physics of Racing, Part 13: Transients

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Obviously, handling is extremely important in any racing car. In an autocross car, it is critical. A poorly handling car with lots of power will not do well at all on the typical autocross course. A Miata or CRX can usually beat a 60's muscle car like a Pontiac GTO even though the Goat may have four or five times the power. Those cars, while magnificently powerful, were designed for straight-line acceleration at the expense of cornering.

This month, we examine one aspect of handling, that of handling *transient* or short-lived forces. Usually, in motor sports contexts, the word "transient" means short-lived cornering forces as opposed to braking and accelerating forces. In braoder contexts, it means any short-lived force.

Transients figure prominently in autocross. Perhaps the epitome of a transient-producing autocross feature is a slalom, which requires a car and driver to flick quickly from left to right and back again. Many courses also feature esses, lane changes, chicanes (dual lane changes), alternating gates, and other variations on the theme. All of these require quick cornering response to transients. Some sports cars, like Elans, MR2's, and X1/9's, are designed specifically to have such quick response. The general rule is that these kinds of cars get you into a corner more quickly than do other kinds. They achieve their response with low weight and low *polar moment of inertia* (PMI). A chief goal of this article is to explain PMI.

Most engineering designs are trade-offs, and designing for quick transient response is no exception. Light weight means, generally, a small engine. Low PMI means, generally, placing the engine as close to the center of mass (CM) as possible. So, many quick-response cars are mid-engined, further constraining engine size. With engine size, we get into another trade-off area: cost versus power. Smaller engines are, generally, less powerful. The cheapest way to get engine power is with size. A big, sloppy, over-the-counter American V8 can cheaply give you 300-400 ft-lb of torque. Getting the same torque from a 1.6 liter four-banger can be very expensive and will put you firmly in the Prepared or Modified ranks. But, a bigger engine is a heavier engine, and that means a beefier (heavier) frame and suspension to support it. Therefore, the cheap way to high torque requires sacrificing some transient response for power. This design approach is typified by Corvettes and Camaros. The general rule is that these kinds of cars get you out of a corner more quickly because of the engine torque.

So, we can divide the sports car universe into the lightweight, quick-response style camp and the ground-thumping, stump-pulling style camp. Some cars straddle the boundary and try to be lightweight, with low PMI, and powerful. These cars are usually very expensive because the fundamental design compromises are pushed with exotic materials and great amounts of engineer time. Ordinary cars are usually mostly one or the other. No one can say which design style is "better." Both kinds of car are great fun to drive. There will be some courses on which quick-response type cars will have top times and others on which the V8's will be unbeatable. Fortunately, these two styles of cars are usually in different classes.

Let's back up that discussion with some physics. What is transient response and how does it relate to polar moment of inertia?

Any object resists a change in its state of motion. If it is not moving, it resists moving. If it is moving, it resists stopping or changing direction. The resistance is generically called *inertia.* With straight line motion, inertia has only one aspect: *mass.* Handling is mostly about cornering, however, not about straight-line motion.

Cornering is a change in the direction of motion of a car. In order to change the direction of motion, we must change the direction in which the car is pointing. To do that, we must rotate or *yaw* the car. However, the car will resist yawing because the various parts of the car will resist changing their states of motion. Let's say we are cornering to the right, hence yawing clockwise. The suspension parts and frame and cables and engine *etc. etc.* in the front part of the car will resist veering to the right off their prior straight-line course and the suspension parts and frame and differential and gas tank *etc. etc.* in the rear will resist veering to the left off their prior straight-line course. From this observation, we can 'package' the inertial resistance to yawing of any car into a convenient quantity, the PMI. What follows is a simplified, two-dimensional analysis. The full, three-dimensional case is conceptuallly similar though more complicated mathematically.

It turns out that the general motion of any large object can be broken up into the motion of the center of mass, treated as a small particle, and the rotation of the object about its center of mass. This means that to do dynamical calculations that account for cornering, we must apply Newton's Second Law, $F = ma$, *twice.* First, we apply the law to all masses in the car taken as an aggregate with their positions measured with respect to a fixed point on the ground. Second, we apply the law individually to the massive parts of the car with their positions measured from the CM in the car while it moves.

Let's make a list of all the N parts in the car. Let the variable i run over all the items in the list; let the masses of the parts be m_i, their positions on the X axis of the ground coordinate grid be x_i and their positions on the Y axis be y_i. We summarize the position information with *vector* notation, writing a bold character, $\boldsymbol{r}_i$, for the position of the i-th part. Vector notation saves us from having to write two (or three) sets of equations, one for each coordinate direction on the grid. For many purposes, a vector can be treated like a number in symbolic arithmetic. We must break a vector equation apart into its constituent *component* equations when it's time to do number-crunching.

The (vector) position $\boldsymbol{R}$ of the CM with respect to the ground is just the mass-weighted average over all the parts of the car:

$$\boldsymbol{R} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{M = \sum_{i=1}^{N} m_i} \tag{1}$$

The external forces on the car are also vectors: they have X components and Y components. So, we write the sum of all the forces on the car with a bold $\boldsymbol{F}$. Similarly, the velocity of the CM is a vector. It is the change in $\boldsymbol{R}$ over a small time, dt, divided by the time. This is written

$$\boldsymbol{V} = \frac{d\boldsymbol{R}}{dt} \tag{2}$$

The d/dt notation is called a *derivative.* In turn, the acceleration is a small change in the velocity divided by the time:

$$\boldsymbol{A} = \frac{d\boldsymbol{V}}{dt} = \frac{d^2\boldsymbol{R}}{dt^2} \tag{3}$$

The d^2/dt^2 notation is called a *second derivative* and results from two derivatives in succession.

Newton's Second Law for the CM of the car is then

$$\boldsymbol{F} = M\frac{d^2\boldsymbol{R}}{dt^2} \tag{4}$$

where M is the total mass of all the parts in the car. Simple, eh? This is a *differential equation*, and theoretical physics is overwhelmingly concerned with the solutions of such things. In this case, a solution is finding $\boldsymbol{R}$ given M and $\boldsymbol{F}$. We can also simplify the writing of the equations in general by replacing time-derivative notations with dots: one dot for one time derivative and two dots for two derivatives. We get

$$\boldsymbol{F} = M\ddot{\boldsymbol{R}} \tag{5}$$

Now, we consider the parts of the car separately as they yaw (and pitch and roll) about the CM while remaining firmly attached to the car . Let's write all position variables measured with respect to the coordinate grid fixed in the car with overbars, so the vector position of the i-th mass in our list is $\overline{\boldsymbol{r}}_i$.

However, we don't need to use vectors (in two dimensions), because in pure yawing motion about the CM of the car, the radial distance of each car part from the CM remains fixed and each part has the same yaw angle as the whole car.

Let the yaw angle of the car and its coordinate grid measured against the ground-based, inertial coordinates be θ. As each car part is affected by forces, it moves in a yaw-arc around the CM. A small amount of yaw is written $d\theta$. Each part moves perpendicularly to a line drawn from the part to the CM of the car, and the distance it moves is equal to its radial distance from the CM, $\overline{r}_i$ (nonbold: a number, not a vector), times the little amount of yaw $d\theta$. Divide by the little time over which the motions are measured, and you have the velocity of each car part:

$$\overline{v}_i = \overline{r}_i \frac{d\theta}{dt} = \overline{r}_i \dot{\theta} \tag{6}$$

Now, it's easy to apply Newton's second law. Equate the force on the i-th part, $\overline{F}_i$, to the mass of the part times the acceleration of the part:

$$\overline{F}_i = m_i \overline{r}_i \ddot{\theta} \tag{7}$$

We're almost done with the math, so hang in there. If we multiply equation 7 by $\overline{r}_i$ on both sides, the left-hand side becomes the torque of the forces on the i-th part about the CM:

$$\overline{\Lambda}_i = \overline{r}_i \overline{F}_i = m_i \overline{r}_i^2 \ddot{\theta} \tag{8}$$

Now, if we sum this equation up over all the parts in our list, we can drop the i subscript:

$$\overline{\Lambda} = \left(\sum_{i=1}^{N} m_i \overline{r}_i^2 \right) \ddot{\theta} \tag{9}$$

remembering that all parts have the same $\ddot{\theta}$. The reason for doing this is that resulting equation *looks like* Newton's Second Law, equation 5. If you replace $\sum m_i \overline{r}_i^2$ with a symbol, $\overline{I}$, the equation is identical in form:

$$\overline{\Lambda} = \overline{I} \ddot{\theta} \tag{10}$$

Physicists like to find formal equivalences amongst equations because they can use the same mathematical techniques to solve all of the them. The equivalences also hints at deeper insights into similarities in the Universe.

Ok, if you haven't already guessed it, $\overline{I} = \sum m_i \overline{r}_i^2$ is the polar moment of inertia. To compute it for a given car, we take all the parts in the car, measure their masses and their distances from the CM, square, multiply and add. In practice, this is very difficult. I doubt if PMIs are measured very often, but when they are, it is probably done experimentally: by subjecting the car to known torques and measuring how quickly yaw angle accumulates.

We can also see that, for a given rotational torque, the acceleration of yaw angle is inversely proportional to $\overline{I}$. Thus, we have backed up, from first principles, our statement that cars with low PMI respond more quickly, by yawing, to transient cornering forces than do cars with large PMI. A car with a low PMI is designed so that the heavy parts—primarily the engine—are as close to the CM as possible. Moving the engine even a couple of inches closer to the CM can dramatically decrease the PMI because it varies as the *square* of the distance of parts from the CM. Since equation 10 is formally equivalent to Newton's Second Law, an analogous insight applies to that Law. A car with low mass responds more quickly to forces with straight-line changes in motion just as a car with low PMI responds more quickly to torques with rotational changes in motion.

Why would one design a car with a high PMI? Only to get a big, powerful engine into it that might have to be placed in the front or the rear, far from the CM. So, take your pick. Choose a car with a low PMI that yaws very quickly and give up on some engine power. Or, choose a car with a colossal engine and give up on some handling quickness.